

# Continuum Fatigue Damage Modeling for Critical Design, Control, and Fault Prognosis

Carl F. Lorenzo  
*Lewis Research Center*  
*Cleveland, Ohio*

Prepared for the  
Sixth International Symposium on Transport Phenomena and  
Dynamics of Rotating Machinery  
cosponsored by the Pacific Center of Thermal Fluids Engineering  
and the U.S. Turbo and Power Machinery Research Center  
Honolulu, Hawaii, February 25-29, 1996



National Aeronautics and  
Space Administration

LIBRARY COPY

JUN 30 1996

LANGLEY RESEARCH CENTER  
LIBRARY NASA  
HAMPTON, VIRGINIA

## CONTINUUM FATIGUE DAMAGE MODELING FOR CRITICAL DESIGN, CONTROL, AND FAULT PROGNOSIS

Carl F. Lorenzo  
National Aeronautics and Space Administration  
Lewis Research Center  
Cleveland, Ohio 44135

### Abstract

This paper develops a simplified continuum (continuous with respect to time, stress, etc.) fatigue damage model for use in critical design, Life Extending Control and fault prognosis. The work is based on the local strain cyclic damage modeling method. New nonlinear explicit equation forms of cyclic damage in terms of stress amplitude are derived to facilitate the continuum modelling. Stress based continuum models are derived. Extension to plastic strain—strain rate models is also presented. Progress toward a non-zero mean stress based is presented. Also new nonlinear explicit equation forms in terms of stress amplitude are derived for this case. Application of the various models to design, control, and fault prognosis is considered.

### I. Introduction

A strong motivation for the current activities to develop continuum fatigue damage models comes from the ongoing work in the areas of life extending and or damage mitigating controls (refs. 1 to 3, respectively). These controls studies seek to create control methodologies to allow the reduction of damage in critical components in aerospace systems by the manner in which the control moves the system transiently between setpoints. The transient damage for critical components in rocket engines has been shown to be capable of reductions on the order of 2/3, by the manner in which the control moves the engine through the transient. This has been accomplished without significant loss in dynamic response (ref. 4). The results quoted above are based on open-loop studies which have been accomplished through the use of gross nonlinear optimization. The continued development of life-extending control requires a damage model which is continuum based, as opposed to current fatigue damage models which are cyclic extrema based. Contemporary cyclic methods require a completed stress strain cycle before the associated damage can be determined. What is required for controls is the ability to predict for the next increment, or continuum, of stress or strain what the associated fatigue damage will be. That is the thrust of this paper. Clearly it is desirable to create as simple a continuum model as possible, since this will allow a more broad application of the life extending control concepts. Only very limited work has been done in this area. The initial work done by A. Ray et al. at Penn State University (ref. 5) has created a useable continuum model for fatigue damage. However, this model is hindered by the requirement of identifying the cycle extrema and then calculating the damage between the extrema based on extrema information. This complicates considerably the use of such a model in a practical control design. It requires a continuous accounting of the cycle extrema and the changing (bookkeeping) of these extrema as the physics of the process progresses. An approach which would use only local stress or strain to infer damage would be simpler and hence superior.

This paper seeks to create a zero mean stress continuum fatigue damage model in two forms, the first form is stress based and the second is plastic strain and strain rate based. The paper introduces new simplified forms for the cyclic damage results, for the zero mean stress case. Progress on a non zero mean stress continuum model is also shown and the open issues that remain in this area are discussed.

The basic objective of this work is to generate a damage model for the fatigue failure of metallic materials which is continuum or differential based as opposed to current theory which is cycle based. The fundamental approach of the paper is to use results from the cyclic local strain method as a basis for the development of the continuum model. The next section will present a short summary on those parts of the local strain approach required for the material which follows. The work of Dowling et al. (ref. 6) will be the focal point that will be used in this paper as the basis of the local strain method. This paper expands on the work in reference 7 in particular with regard to applications.

## Local Strain Method

The basis of current damage (fatigue/fracture) approaches, study the experimental results of applying cyclic loads of various amplitude and various mean (constant) bias loads, and summarize/generalize these to allow prediction of arbitrary combinations of loading cycles. Many variations and methods have been evolved. A fairly straight forward approach (called the local strain approach) by Dowling et al. (ref. 6) will be the foundation for the analysis which follows.

A typical stress-strain hysteresis loop is shown in figure 1(a). The effect of cycle amplitude changes (with zero mean stress) is illustrated in figure 1(b). The back bone plot (cyclic stress-strain curve) which is the locus of the extrema of the stress-strain cycles is shown (ref. 6) to have the mathematical form (for materials of interest);

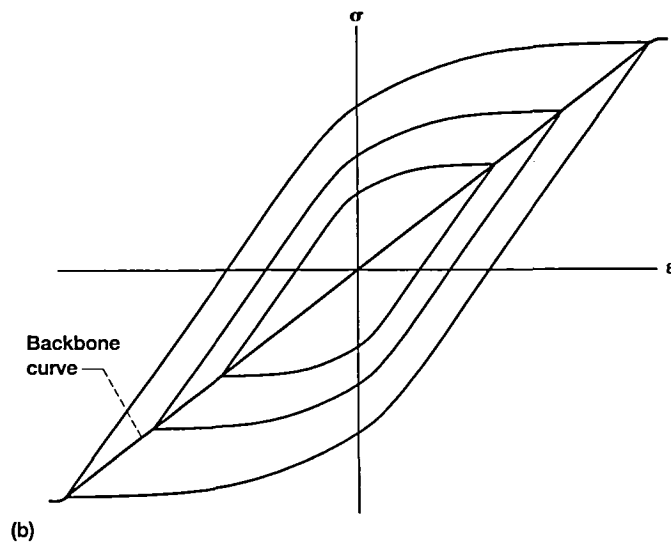
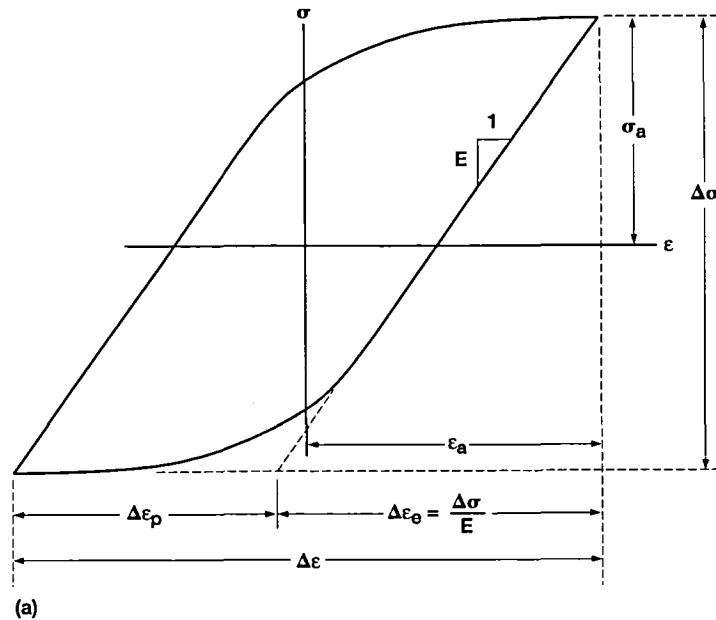


Figure 1.—Cyclic stress-strain. (a) Typical hysteresis loop. (b) Stress-strain cycles of various amplitudes—with backbone curve.

$$\epsilon_a = \frac{\sigma_a}{E} + \left( \frac{\sigma_a}{A} \right)^{1/s} \quad (1)$$

where  $\epsilon_a = \Delta\epsilon/2$  and  $\sigma_a = \Delta\sigma/2$  are strain and stress amplitude, respectively, and  $E$ ,  $A$ , and  $s$  are material constants. The cyclic damage associated with repeated hysteresis stress-strain cycles of a given amplitude is determined by experimental observation to have the mathematical form:

$$\epsilon_a = \frac{\sigma'_f}{E} (2N_f)^b + \epsilon'_f (2N_f)^c \quad (2)$$

where  $b$ ,  $c$ ,  $E$ ,  $\sigma'_f$  and  $\epsilon'_f$  are constants for a particular material and  $N_f$  is the number of cycles to failure. The plot of figure 2 shows a typical curve as described by equation (2).

The total strain amplitude (eq. (1)) is seen to be composed of two parts the elastic strain contribution.

$$\epsilon_{a_e} = \frac{\sigma_a}{E} = \frac{\sigma'_f}{E} (2N_f)^b \quad (3)$$

and the plastic strain contribution

$$\epsilon_{a_p} = \left( \frac{\sigma_a}{A} \right)^{1/s} = \epsilon'_f (2N_f)^c \quad (4)$$

Equation (2) now can be used to estimate the damage associated with a cycle of strain of amplitude  $\epsilon_a$ . Since  $N_f$  is the number of cycles to failure at amplitude  $\epsilon_a$  then  $1/N_f$  is the damage of a single cycle (assuming no effect of accumulated damage). Therefore  $n$  cycles of amplitude  $\sigma_a$  will create a damage  $D$  of

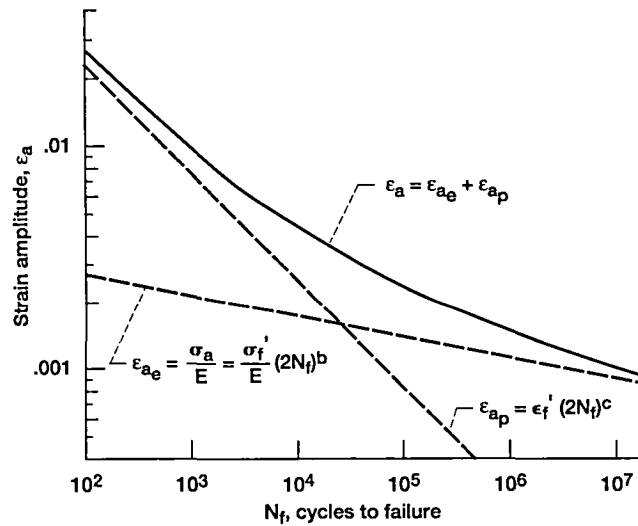


Figure 2.—Cycles to failure versus strain amplitude.

$$D = \frac{n}{N_f} \quad (5)$$

where  $D = 1$  represents failure. Then the Palmgren-Miner equation may be used to determine the damage for cycles of different amplitudes, i.e.

$$D = \sum_i \frac{n_i}{N_{fi}} \quad (6)$$

where  $i$  represents the various amplitudes composing the strain history. Various corrections can be applied for the effect of mean stress superimposed on the cycles and alternate methods have been evolved to account for the effect of damage accumulation (nonlinearity) on the damage of any particular cycle.

#### Zero Mean Stress Continuum Damage Model

Because of the strong nonlinear terms in equations (1) and (2) above, these equation forms are not directly suitable for the continuum model development. Fortunately, there is a redundancy in the material properties ( $b, c, E, \sigma'_f, \epsilon'_f$ , and  $s$ ) that can be used. This can be determined from equations (3) and (4) by solving for  $2N_f$  in each and eliminating  $2N_f$ , Thus

$$\epsilon_{p_a} = \epsilon'_f \left( \frac{\sigma_a}{\sigma'_f} \right)^{c/b} \quad (7)$$

where  $\epsilon_{p_a}$  is the amplitude of the plastic strain.

Now using  $(\sigma_a/A)^{1/s}$  to eliminate  $\epsilon_{p_a}$  and rewriting the right hand side of equation (7) gives

$$\left( \frac{\sigma_a}{A} \right)^{1/s} = \left[ \frac{\sigma_a}{\sigma'_f} (\epsilon'_f)^{b/c} \right]^{c/b} \quad (8)$$

For this to be true the following must apply

$$\frac{1}{s} = \frac{c}{b} \quad (9)$$

and

$$A \epsilon_f'^{b/c} = A \epsilon_f'^s = \sigma'_f \quad (10)$$

For the materials study in reference 6, these relationships are found to be virtually exact for RQC-100 and approximately correct for MAN-TEN steel. These equations are assumed to be generally approximately correct for many metallic materials and will be used in the analysis which follows.

In the analysis that follows stress,  $\sigma$ , will be considered as the independent variable. While it is true that in typical fatigue testing the strain amplitude is held constant, it also seems clear by analogy to fluid and current flow and other physical processes that load or potential (stress) is the cause of motion (strain in this case). Therefore, equations (1) and (2) will be combined eliminating the strain amplitude  $\epsilon_a$ , thus

$$\epsilon_a = \frac{\sigma_a}{E} + \left( \frac{\sigma_a}{A} \right)^{1/s} = \frac{\sigma'_f}{E} (2N_f)^b + \epsilon'_f (2N_f)^c \quad (11)$$

or since  $\delta_{cyc} = 1/N_f$ , where  $\delta_{cyc}$  is the damage per cycle this can be written as

$$\epsilon_a = \frac{\sigma_a}{E} + \left( \frac{\sigma_a}{A} \right)^{1/s} = \frac{\sigma'_f}{E} \left( \frac{\delta_{cyc}}{2} \right)^{-b} + \epsilon'_f \left( \frac{\delta_{cyc}}{2} \right)^{-c} \quad (12)$$

The elastic terms and the plastic terms in equation (12) are now considered separately to determine  $\delta_{cyc}$ . For the elastic terms,

$$\frac{\sigma_a}{E} = \frac{\sigma'_f}{E} \left( \frac{\delta_{cyc_e}}{2} \right)^{-b} \quad (13a)$$

yields

$$\delta_{cyc_e} = 2 \left( \frac{\sigma_a}{\sigma'_f} \right)^{-1/b} \quad (13b)$$

For the plastic terms

$$\left( \frac{\sigma_a}{A} \right)^{1/s} = \epsilon'_f \left( \frac{\delta_{cyc_p}}{2} \right)^{-c} \quad (14a)$$

yields

$$\delta_{cyc_p} = 2 \left( \frac{1}{\epsilon'_f} \right)^{-1/c} \left( \frac{\sigma_a}{A} \right)^{-1/cs} = 2 \sigma_a^{-1/cs} \left( \frac{1}{A \epsilon'_f{}^s} \right)^{-1/cs} \quad (14b)$$

Using the material relations of equations (9) and (10), the damage per cycle for the plastic part of equation (12) becomes

$$\delta_{cyc_p} = 2 \left( \frac{\sigma_a}{\sigma'_f} \right)^{-1/b} \quad (14c)$$

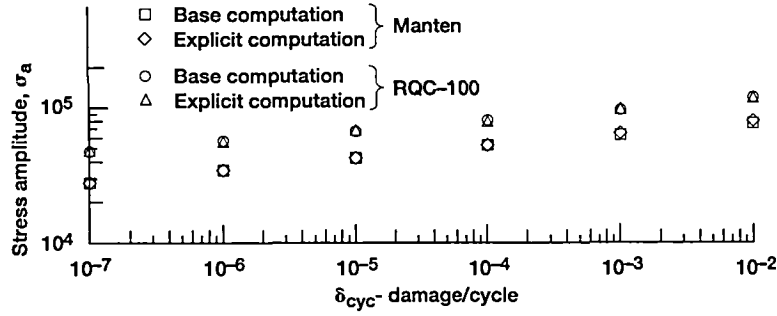


Figure 3.—Damage per cycle,  $\delta_{cyc}$  versus stress amplitude  $\sigma_a$ , for Manten and RQC-100 steels; for base computation (EQ 12) and explicit computation (EQ 15).

Now since equations (13b) and (14c) are the same and satisfy both parts of equation (12), it is the solution (restatement) of equation (12). Thus, the zero mean stress case cyclic damage law can be written.

$$\delta_{cyc} = 2 \left( \frac{\sigma_a}{\sigma'_f} \right)^{-1/b} \quad (15a)$$

or

$$N_f = \frac{1}{2} \left( \frac{\sigma_a}{\sigma'_f} \right)^{1/b} \quad (15b)$$

These equations represent the average (or midlife) damage/cycle in terms of the stress amplitude (at midlife) for strain controlled cyclic fatigue. To validate this result against data, the  $\epsilon_a$  versus  $N_f$  life results for RQC-100 and MANTEN steels (ref. 6) were used as a test case. To do this equation (12) was solved numerically for  $\sigma_a$  versus  $\delta_{cyc}$  and compared to the results of equation (15), figure 3. The results appear to be within the accuracy of the numerical solutions over five decades of  $\delta_{cyc}$ . The explicit form of equation (15) now allows further analysis toward a continuum damage model.

With stress as the independent variable driving damage and knowing the damage per cycle (eq. (15)) then

$$\int_{cycle} \delta'(\sigma) d\sigma = \delta_{cyc} = 2 \left( \frac{\sigma_a}{\sigma'_f} \right)^{-1/b} \quad (16)$$

where  $\delta'(\sigma)$  is the damage rate  $\delta'(\sigma) = d\delta/d\sigma$  as the cycle is transversed. Now a critical question is; over which part of the cycle does the damage occur. Various assumptions are possible, with reference to figure 4. It is plausible that damage is not likely generated during the relaxation (unloading) phases of the cycle, namely  $A \rightarrow B$  and  $C \rightarrow D$ , although cracks and acoustic emissions may be observed. Damage is most likely during tensile stressing  $D \rightarrow A$  and may also occur during compressive stressing  $B \rightarrow C$ .

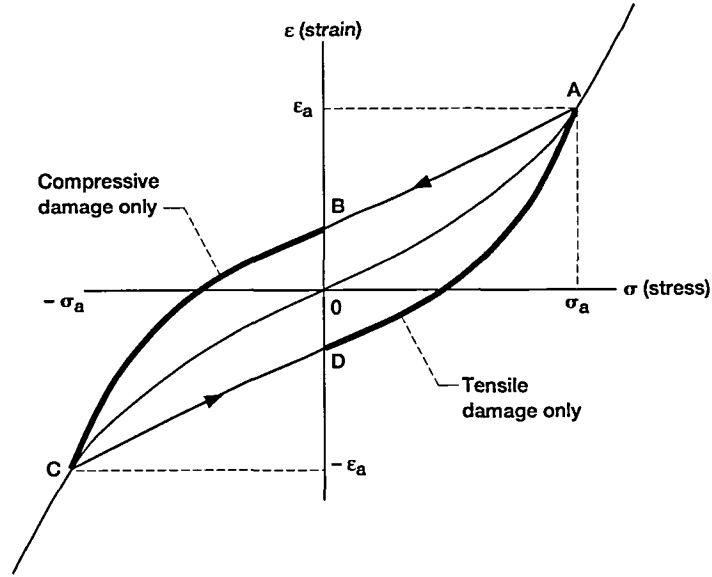


Figure 4.—Assumed damage producing domains.

The initial assumption will be that damage occurs only for  $\sigma > 0$  and  $\sigma$  increasing (domain  $D \rightarrow A$ , fig. 4), again assuming linear damage accumulation then

$$\int_0^{\sigma_a} \delta'(\sigma) d\sigma = \delta_{cyc} = 2 \left( \frac{\sigma_a}{\sigma'_f} \right)^{-1/b} \quad (17)$$

It is readily shown that

$$\delta'(\sigma) = -\frac{2}{b\sigma'_f} \left( \frac{\sigma}{\sigma'_f} \right)^{-(1+b)/b} \quad (18a)$$

or

$$\delta'(\sigma) = \frac{-2}{b\sigma'_f} \left( \frac{|\sigma|}{\sigma'_f} \right)^{-(1+b)/b} \quad (18b)$$

also works. Thus the rate at which damage accumulates over the cycle is given by equation (18). This result is extremely nonlinear, for example for RQC-100 steel,  $b = 0.075$ ,  $\sigma'_f = 1.68 \times 10^5$  is

$$\delta'(\sigma) = 1.5872 \times 10^{-4} \left( \frac{\sigma}{1.68 \times 10^5} \right)^{12.3333} \quad (18d)$$

This form shows the extreme sensitivity to small changes (and errors) in stress. It is important to note that equation (18) is independent of cycle amplitude, i.e., applies to any zero mean stress cycle. Also equation (18) does not



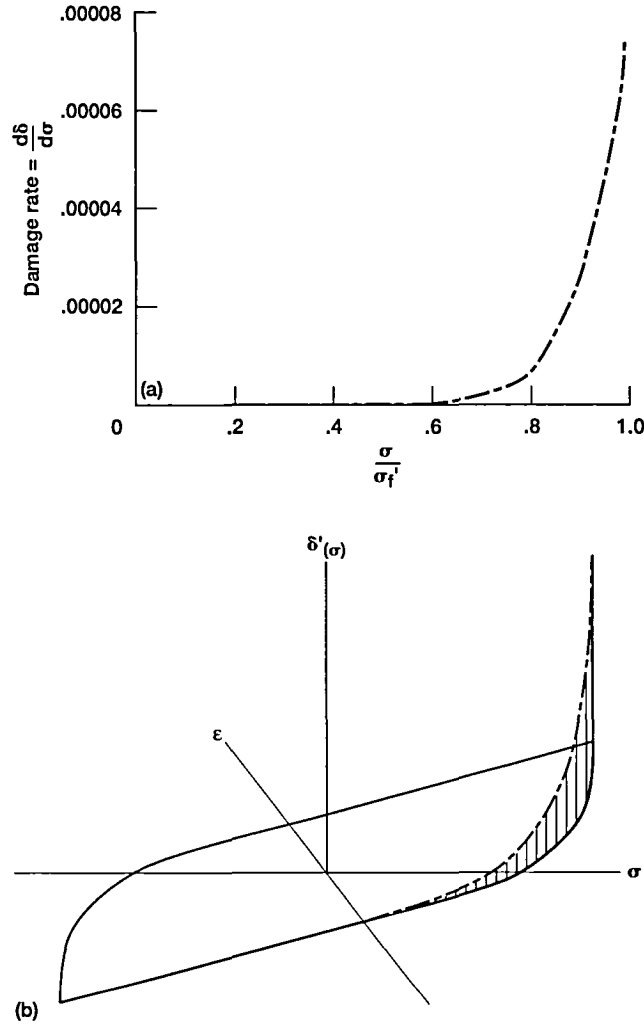


Figure 5.—Damage rate versus stress. (a) RQC-100 steel.  
(b) Location on hysteresis cycle.

depend on knowledge of the cycle extrema (reference values). A plot of equation (18) for RQC-100 steel is presented in figure 5(a). Figure 5(b) shows the damage rate location in a cycle.

If damage occurs equally in tension and compression then

$$\delta'(\sigma) = -\frac{1}{b\sigma'_f} \left( \frac{|\sigma|}{\sigma'_f} \right)^{-(1+b)/b} \quad (19)$$

applies over domain  $B \rightarrow C$  and  $D \rightarrow A$  in figure 4. In view of the mean stress effect on damage, it is more likely that greater damage occurs, over the domain  $D \rightarrow A$  than  $B \rightarrow C$ , therefore an unequal damage distribution such as

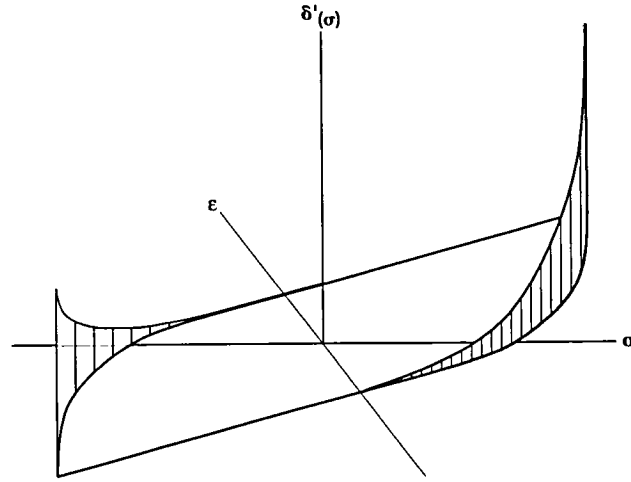


Figure 6.—Damage rate distribution for combined tensile and compressive damage.

$$\delta'(\sigma) = \begin{cases} -\frac{2k}{b\sigma'_f} \left( \frac{|\sigma|}{\sigma'_f} \right)^{-(1+b)/b} & \text{for } \sigma > 0, \sigma \text{ increasing} \\ -\frac{2(1-k)}{b\sigma'_f} \left( \frac{|\sigma|}{\sigma'_f} \right)^{-(1+b)/b} & \text{for } \sigma < 0, \sigma \text{ decreasing} \end{cases} \quad (20)$$

may be a more likely scenario than the equal damage case equation (19). In this equation the parameter  $k$ , weights the tension side damage relative to the compressive side. Figure 6 illustrates the damage distribution for this case.

Any of the above forms (eqs. (18) to (20)), can be used as a basis for damage estimation for Life Extending or Damage Mitigating Control. These forms are particularly useful because they depend only on stress (and material constants), which can be estimated from associated structural models. Clearly the case represented by equation (20), is the most general form containing the other two by proper selection of  $k$ . However the case of equation (18) shows the simplicity of the approach and is easily used to derive further results which may then be generalized.

To convert the above results to a model useable in the time domain, the case with damage occurring only with increasing tensile load (eq. (18)) will be considered.

For this case the damage rate is given by:

$$\delta'(\sigma) = \frac{d\delta}{d\sigma} = -\frac{2}{b\sigma'_f} \left( \frac{\sigma}{\sigma'_f} \right)^{-(1+b)/b} \quad \sigma > 0, \sigma \text{ increasing} \quad (21)$$

In the time domain the damage rate  $D$  will be given as

$$\dot{D}(t) = \frac{d\delta}{d\sigma} \frac{d\sigma}{dt} = -\frac{2}{b\sigma'_f} \left( \frac{\sigma}{\sigma'_f} \right)^{-(1+b)/b} \frac{d\sigma}{dt} \quad \sigma > 0, \sigma \text{ increasing} \quad (22)$$

and the accumulated damage will be

$$D(t) = \int_0^t -\frac{2}{b\sigma'_f} \left( \frac{\sigma}{\sigma'_f} \right)^{-(1+b)/b} \frac{d\sigma}{dt} dt \quad \sigma > 0, \sigma \text{ increasing} \quad (23)$$

Extension to the more general case equation (20) is obvious.

It is also noted that  $D(t)$  is a monotonically increasing function of time.

#### Strain Strain-Rate Continuum Damage Model

The Damage Equation (eq. (18)) in terms of stress is based on an analysis of a hysteresis cycle, under the assumptions of zero mean stress and that all the damage occurs during the extensive (tensile) part of the loop ( $\sigma > 0$  and  $\sigma$  increasing). Extension to compressive damage is obvious and leads to symmetric terms with fractional multipliers (eq. (20)).

The results are believed to be correct so long as hysteresis cycles (or similar load) are being analyzed. However, consider, the loading case shown in figure 7. In this loading scenario the last segment,  $A \rightarrow B$ , completes a hysteresis loop  $B, E, F, A, B$ . But shortly into the return (Point C) the loading is again increased. A profile similar to  $C \rightarrow D$  will be experienced. Note that most of  $C \rightarrow D$  is elastic. After point D is reached, strong plastic strain will be again be experienced.

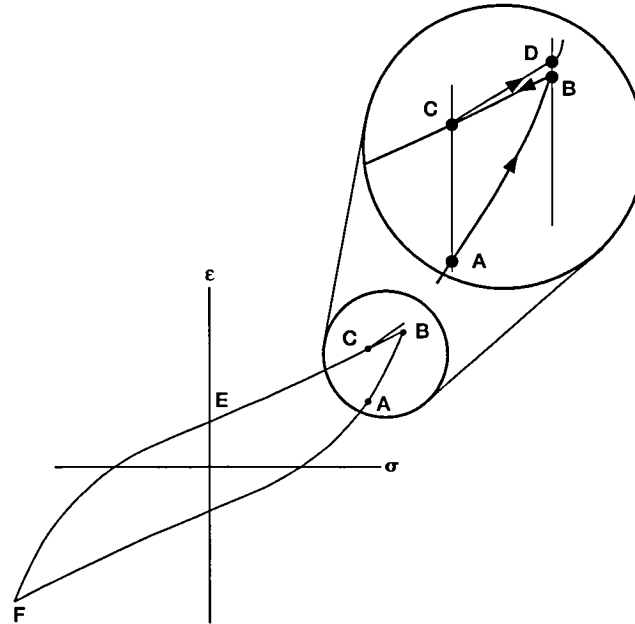


Figure 7.—Illustrative loading case.

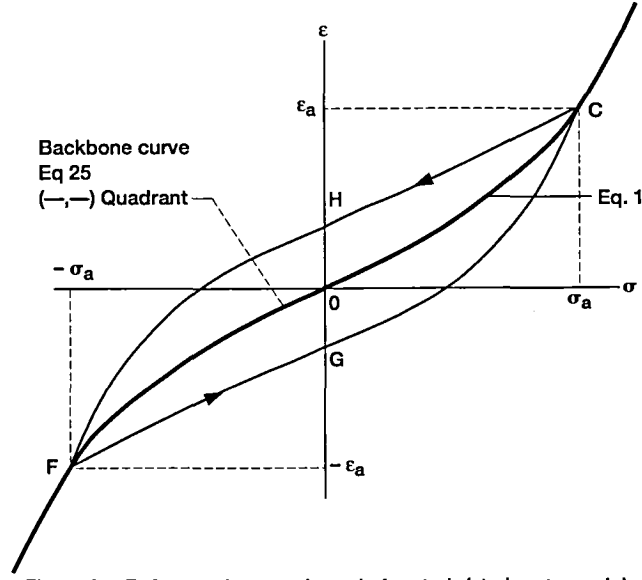


Figure 8.—Reference hysteresis cycle for strain/strain rate model.

Integration of equation (18) will predict the same damage for  $C \rightarrow D$  as  $A \rightarrow B$ , since the stress levels and changes are the same. Use of equation (18) for  $A \rightarrow B$  will yield a correct damage estimate. However, it is believed that its use on leg  $C \rightarrow D$  will be overly conservative since the plastic deformation is relatively much smaller. It is further believed that a damage rate equation based on plastic strain and strain rate will more accurately predict the damage rates on both  $A \rightarrow B$  and  $C \rightarrow D$ , and will be more generally applicable.

The derivation of the damage model in terms of plastic strain and strain rate will be derived for the tensile damage only case and is based therefore on equation (18a).

Consider a hysteresis loop (fig. 8) of amplitude  $\sigma_a$ ,  $\epsilon_a$ . For the case described only the lower curve FGC need be considered. The equation for this curve, reference 6 is given as

$$\frac{\epsilon - \epsilon_r}{2} = \frac{\sigma - \sigma_r}{2E} + \left( \frac{\sigma - \sigma_r}{2A} \right)^{1/s} \quad (24)$$

where  $\sigma$  and  $\epsilon$  are the stress and strain respectively on FGC and  $(\sigma_r, \epsilon_r)$  are the coordinates of the previous strain reversal (point F here). The curve for the locus of reversal points (for (-,-) quadrant) is

$$\epsilon_a = -\frac{\sigma_a}{E} - \left( \frac{\sigma_a}{A} \right)^{1/s} \quad (25)$$

Letting  $\epsilon_r = -\epsilon_a$  and  $\sigma_r = -\sigma_a$  in equation (24) and replacing  $\epsilon_a$  by equation (25) gives

$$\epsilon = \epsilon_p + \epsilon_e = \frac{\sigma}{E} - \left( \frac{\sigma_a}{A} \right)^{1/s} + 2 \left( \frac{\sigma + \sigma_a}{2A} \right)^{1/s} \quad \text{for FGC} \quad (26)$$

where  $\epsilon_p$  is the plastic strain component and  $\epsilon_e$  is the elastic strain component ( $\sigma/E$ ). Considering only the plastic strain component gives

$$\epsilon_p = 2 \left( \frac{\sigma + \sigma_a}{2A} \right)^{1/s} - \left( \frac{\sigma_a}{A} \right)^{1/s} \quad \text{for FGC} \quad (27)$$

Now, differentiating with respect to  $\sigma$  yields the plastic strain rate as

$$\epsilon'_p = \frac{d\epsilon_p}{d\sigma} = \frac{1}{As} \left( \frac{\sigma + \sigma_a}{2A} \right)^{(1-s)/s} \quad \text{for FGC} \quad (28)$$

Solving for  $\sigma_a$  from this equation gives

$$\sigma_a = 2A \left( As\epsilon'_p \right)^{s/(1-s)} - \sigma \quad (29)$$

This equation indicates that given a family of zero mean stress hysteresis curves, of various amplitudes, the particular curve corresponding to stress amplitude,  $\sigma_a$ , can be identified knowing the plastic strain slope  $\epsilon'_p$  and the stress level.

Now  $\sigma_a$  can be replaced in equation (27) by the expression of equation (29) and solving for  $\sigma$  gives

$$\sigma = 2A \left( As\epsilon'_p \right)^{s/(1-s)} - A \left[ 2 \left( As\epsilon'_p \right)^{1/(1-s)} - \epsilon_p \right]^s \quad (30)$$

This equation relates the stress level to the plastic strain rate and strain over curve FGC. The relationship holds regardless of cycle stress amplitude  $\sigma_a$ . The damage rate  $\delta'(\sigma)$  in equation (18) can now be expressed in terms of  $\epsilon_p$  and  $\epsilon'_p$  by substituting equation (30) into equation (18), thus

$$\delta'(\epsilon_p, \epsilon'_p) = -\frac{2}{b\sigma'_f} \left\{ \frac{2A \left( As\epsilon'_p \right)^{s/(1-s)} - A \left[ 2 \left( As\epsilon'_p \right)^{1/(1-s)} - \epsilon_p \right]^s}{\sigma'_f} \right\}^{-(1+b)/b} \quad (31)$$

$\sigma > 0$ ,  $\sigma$  increasing

This is the basic result of this section, it applies to the tensile stress only case. Extension to the compressive damage cases requires consideration of the curve CHF in figure 8. Again reference 6 gives the equation for this curve as

$$\frac{\epsilon_r - \epsilon}{2} = \frac{\sigma_r - \sigma}{2E} + \left( \frac{\sigma_r - \sigma}{2A} \right)^{1/s} \quad (32)$$

With  $\sigma_r = \sigma_a$  and  $\epsilon_r = \epsilon_a = \sigma_a/E + (\sigma_a/A)^{1/s}$  the plastic strain for CHF becomes

$$\epsilon_p = \left(\frac{\sigma_a}{A}\right)^{1/s} - 2\left(\frac{\sigma_a - \sigma}{2A}\right)^{1/s} \quad (33)$$

and

$$\epsilon'_p = \frac{1}{As} \left(\frac{\sigma_a - \sigma}{2A}\right)^{(1-s)/s} \quad \text{for CHF} \quad (34)$$

Following a derivation similar to that above and using equation (19) as the basis gives

$$\delta'(\epsilon_p, \epsilon'_p) = -\frac{2}{b\sigma'_f} \left\{ \frac{\left[ 2A(As\epsilon'_p)^{s/(1-s)} + A\left[ \epsilon_p + 2(As\epsilon'_p)^{1/(1-s)} \right]^s \right]}{\sigma'_f} \right\}^{-(1+b)/b} \quad (35)$$

$\sigma < 0$ ,  $\sigma$  decreasing

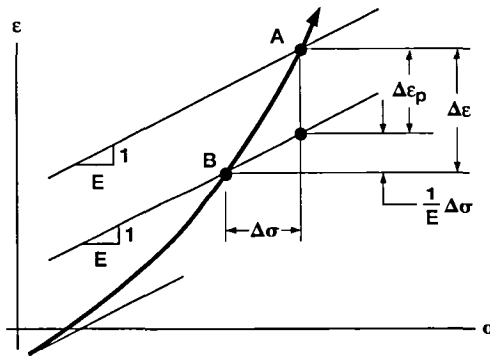
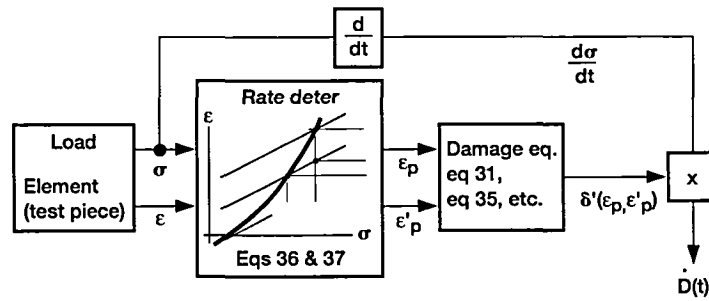


Figure 9.—Damage computation based on plastic strain and plastic strain rate.

as the compressive damage only expression. Extension to the combined tensile compressive damage case of equation merely requires the multiplication of equations (31) and (35) by  $k$  and  $(1 - k)$  respectively.

The discussion that follows shows how  $\delta'(\epsilon_p, \epsilon'_p)$  can be determined in application. It will be assumed stress,  $\sigma$ , and strain,  $\epsilon$ , measurements or estimates are available at the critical load (damage) point(s) of the structure. It is desired to estimate  $\epsilon_p(t)$  and  $\epsilon'_p(t)$  at such a point. The estimate(s) are based on the graph of figure 9. The Elastic Modulus ( $E$ ) is assumed to be known and constant. During elastic straining (loading)

$$\epsilon_e = \frac{\sigma}{E}$$

Then the plastic strain at point A, is given by

$$\epsilon_{pA} = \epsilon_A - \frac{\sigma_A}{E} \quad (36)$$

The plastic strain rate  $\epsilon'_{pA}$  is determined based on a small step from point B, thus:

$$\epsilon'_p = \left. \frac{d\epsilon_p}{d\sigma} \right|_A \approx \frac{\Delta\epsilon_{pA}}{\Delta\sigma_A} = \frac{\epsilon_A - \epsilon_B - \frac{1}{E}(\sigma_A - \sigma_B)}{\sigma_A - \sigma_B} \quad (37a)$$

or

$$\epsilon'_{pA} \approx \frac{\epsilon_A - \epsilon_B}{\sigma_A - \sigma_B} - \frac{1}{E} \quad (37b)$$

This form is valid for both legs of the hysteresis cycle so long as  $B$  is taken as the trailing point.

Now of course equations (31) and (35) are converted to time dependent forms as was done in equation (22) i.e.,

$$\dot{D}(t) = \frac{d\delta}{d\sigma} \frac{d\sigma}{dt} = \delta'(\epsilon_p, \epsilon'_p) \frac{d\sigma}{dt} \quad (38)$$

#### Non-Zero Mean Stress Continuum Damage Model

The previous sections have suggested continuous models based on the case of zero mean stress. It is important to examine the effect of mean stress in this regard. While many expressions have been generated in the cyclic damage format for the effect of mean stress, the following equation (ref. 8) will be used to attempt to generate a continuum nonzero mean stress damage model:

$$\epsilon_a = \frac{\sigma'_f - \sigma_m}{E} (2N_f)^b + \epsilon'_f \left( 1 - \frac{\sigma_m}{\sigma'_f} \right)^{c/b} (2N_f)^c \quad (39)$$

In this expression  $\sigma_m$  represents the mean cyclic stress. Assuming the cyclic stress-strain behavior of the material is not altered by mean stress then

$$\epsilon_a = \frac{\sigma_a}{E} + \left( \frac{\sigma_a}{A} \right)^{1/s} \quad (40)$$

and  $N_f = 1/\delta_{cyc}$  may be used to express equation (39) in terms of stress amplitude and damage per cycle giving

$$\frac{\sigma_a}{E} + \left( \frac{\sigma_a}{A} \right)^{1/s} = \frac{\sigma'_f - \sigma_m}{E} \left( \frac{\delta_{cyc}}{2} \right)^{-b} + \epsilon'_f \left( 1 - \frac{\sigma_m}{\sigma'_f} \right)^{c/b} \left( \frac{\delta_{cyc}}{2} \right)^{-c} \quad (41)$$

Equating the elastic parts and solving for  $\delta_{cyc}$  yields

$$\delta_{cyc_e} = 2 \left( \frac{\sigma_a}{\sigma'_f - \sigma_m} \right)^{-1/b} \quad (42)$$

Equating the plastic parts of equation (41) gives

$$\left( \frac{\sigma_a}{A} \right)^{1/s} = \epsilon'_f \left( 1 - \frac{\sigma_m}{\sigma'_f} \right)^{c/b} \left( \frac{\delta_{cyc_p}}{2} \right)^{-c} \quad (43)$$

or

$$\left( \frac{\delta_{cyc_p}}{2} \right)^{-c} = \frac{1}{\epsilon'_f} \left( \frac{\sigma_a}{A} \right)^{1/s} \left( \frac{\sigma'_f - \sigma_m}{\sigma'_f} \right)^{-c/b} \quad (44)$$

Using the material relationships, equations (9) and (10) and after some algebraic manipulation, solving for  $\delta_{cyc_p}$  yields amazingly

$$\delta_{cyc_p} = 2 \left( \frac{\sigma_a}{\sigma'_f - \sigma_m} \right)^{-1/b} \quad (45)$$

The logic proceeds as in the zero mean stress case, i.e., since

$$\delta_{cyc} = 2 \left( \frac{\sigma_a}{\sigma'_f - \sigma_m} \right)^{-1/b} \quad (46)$$

satisfies both elastic and plastic terms of equation (41) it is a solution of equation (41) and represents a simplified mean stress cyclic damage law. The value of this form (eq. (46)) over that of equations (41) or (39) is that it may be



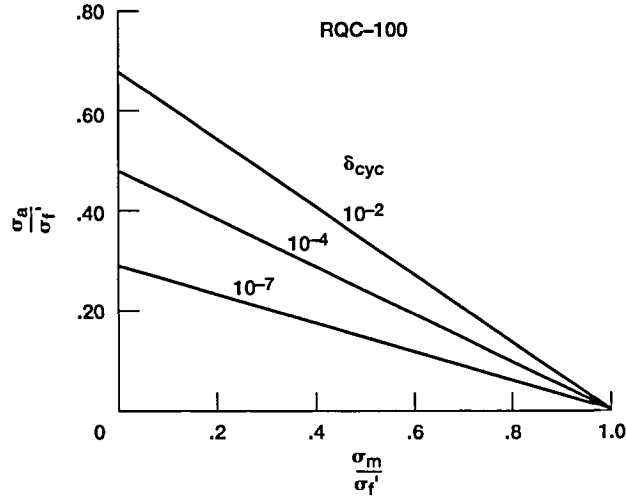


Figure 10.—Effects of mean stress and stress amplitude on damage per cycle,  $\delta_{cyc}$ , for RQC-100 steel.

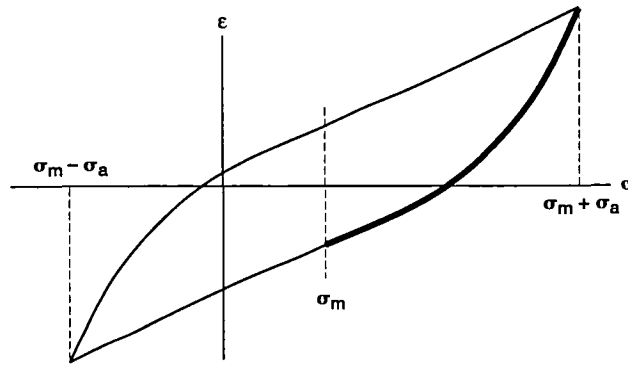


Figure 11.—Assumed damage location for tensile side only mean stress damage model.

explicitly solved for any quantity. Figure 10 shows the character of equation (46) for various levels of damage/cycle,  $\delta_{cyc}$ .

With the availability of the explicit form (eq. (46)) a mean stress condition continuum damage model formulation may be attempted.

The case where all the cyclic damage occurs only between  $\sigma_m$  and  $\sigma_m + \sigma_a$  on the tensile leg (fig. 11) is considered. The problem is to determine a damage rate  $\delta'(\sigma)$  over this domain such that:

$$\int_{\sigma_m}^{\sigma_m + \sigma_a} \delta'(\sigma) d\sigma = \delta_{cyc} = 2 \left( \frac{\sigma_a}{\sigma_f' - \sigma_m} \right)^{-1/b} \quad (47)$$

Now since

$$\int_a^x f(y) dy = \int_0^{x-a} f(y+a) dy$$

the problem can be reformulated as

$$\int_0^{\sigma_a} \delta'(\sigma + \sigma_m) d\sigma = 2 \left( \frac{\sigma_a}{\sigma'_f - \sigma_m} \right)^{-1/b} \quad (48)$$

Then differentiating the right-hand side of equation (48) gives

$$\delta'(\sigma + \sigma_m) = -\frac{2}{b} \left( \frac{1}{\sigma'_f - \sigma_m} \right)^{-1/b} (\sigma)^{-(1/b)-1}$$

and the desired integrand then is given by

$$\delta'(\sigma) = -\frac{2}{b} \left( \frac{1}{\sigma'_f - \sigma_m} \right)^{-1/b} (\sigma - \sigma_m)^{-(1+b)/b} \quad \text{for } \sigma'_f \geq \sigma \geq \sigma_m \geq 0, \sigma \text{ increasing} \quad (49)$$

This then (eq. (49)), is the desired damage rate for a hysteresis cycle with a mean stress condition. The hope prior to starting this part of the analysis was that this expression would be found to be independent of  $\sigma_m$  so that a “universal” stress based continuum form would be obtained, this unfortunately was not the case.

This equation (eq. (49)) parallels equation (18) and can be seen to reduce to it when  $\sigma_m = 0$ . Further, if it assumed that damage occurs equally in tension increasing and compression increasing modes, then

$$\delta'(\sigma) = -\frac{1}{b} \left( \frac{1}{\sigma'_f - \sigma_m} \right)^{-1/b} (\|\sigma - \sigma_m\|)^{-(1+b)/b} \quad (50)$$

may be used, paralleling equation (19). And for unequal damage on the tensile and compressive legs for the mean stress case

$$\delta'(\sigma) = \begin{cases} -2 \frac{k}{b} \left( \frac{1}{\sigma'_f - \sigma_m} \right)^{-1/b} (\|\sigma - \sigma_m\|)^{-(1+b)/b} & \sigma > \sigma_m, \sigma \text{ increasing} \\ -2 \frac{(1-k)}{b} \left( \frac{1}{\sigma'_f - \sigma_m} \right)^{-1/b} (\|\sigma - \sigma_m\|)^{-(1+b)/b} & \sigma < \sigma_m, \sigma \text{ decreasing} \end{cases} \quad (51)$$

and equation (51) parallels equation (20).

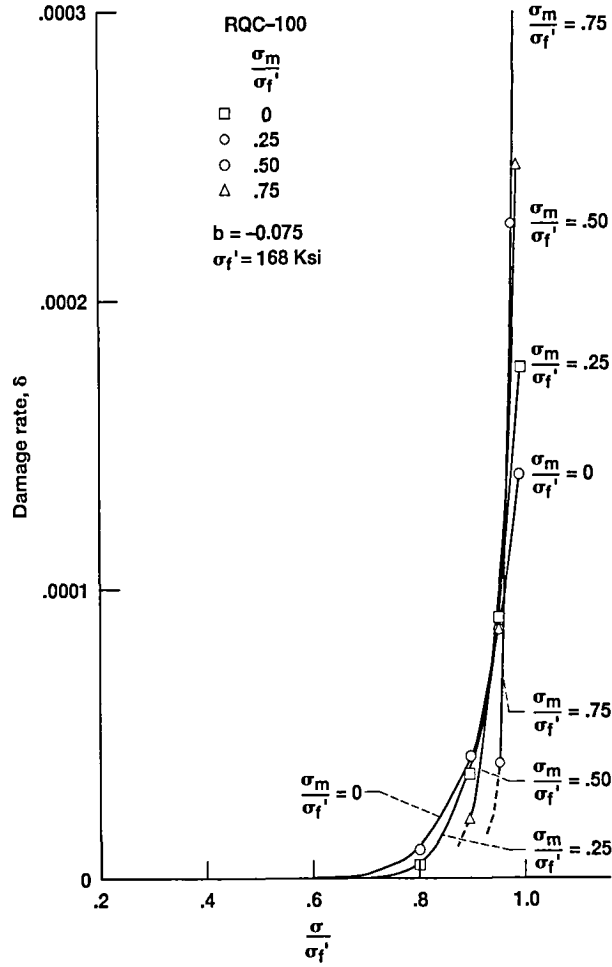


Figure 12.—Damage rate versus normalized stress level with  $\sigma_m/\sigma_f' = 0, .25, .50, .75$ .

The effect of mean stress level on damage rate for RQC-100 steel is shown in figure 12 for the tensile only damage assumption (eq. (49)). It is very important to note here that for stress levels less than 0.925 the zero mean stress damage rate is greater than all non zero mean stress cases (for  $\sigma_m = 0$ ). That is, the simple zero mean stress damage law of equation (18), is conservative in predicting damage rate at these stress levels (with mean stress). This suggests that it may be feasible to use the envelope (maximum) damage curve as a conservative damage law for practical Life Extending Control applications.

For  $\sigma_m > 0$ , the maximum damage curve at a given (constant) stress level, is determined by setting the derivative of  $\delta'(\sigma)$  in equation (49) to zero, thus

$$\frac{d\delta'}{d\sigma_m} = \frac{2}{b^2}(\sigma - \sigma_m)^{-(1+b)/b}(\sigma_f' - \sigma_m)^{-(1+b)/b} - \frac{2(1+b)}{b^2}(\sigma_f' - \sigma_m)^{1/b}(\sigma - \sigma_m)^{-(1+2b)/b} = 0 \quad (52)$$

After considerable algebra, this yields the condition

$$\frac{\sigma_m}{\sigma_f'} = -\frac{1}{b} \frac{\sigma}{\sigma_f'} + \frac{1+b}{b} \quad \text{for } \sigma_f' > \sigma_m > 0 \quad (53)$$

Substituting this results into equation (49) and after considerable simplification

$$\delta'(\sigma)_{max} = \frac{2(1+b)^{-(1+b)/b}}{\sigma'_f - \sigma} \quad \text{for } \sigma'_f > \sigma_m > 0 \quad (54)$$

In words; the maximum damage rate is inversely proportional to the distance of the stress from  $\sigma'_f$ . It can be shown from equation (53), with  $\sigma_m / \sigma'_f = 0$  that equation (54) only applies for

$$\frac{\sigma}{\sigma'_f} \geq 1 + b \quad (55)$$

Thus for RQC-100,  $1 + b = 0.925$ , and equation (54) is seen to determine  $\delta'_{max}$  for  $\sigma / \sigma'_f \geq 0.925$  while below that level the zero mean stress equation (18) dominates. Extension, of the mean stress based damage laws to the time domain follows that shown for equations (22) and (23) with obvious changes in  $d\delta/d\sigma$ . The plot of damage rate versus normalized stress level (fig. 13) compares the mean stress results with the maximum damage rate result

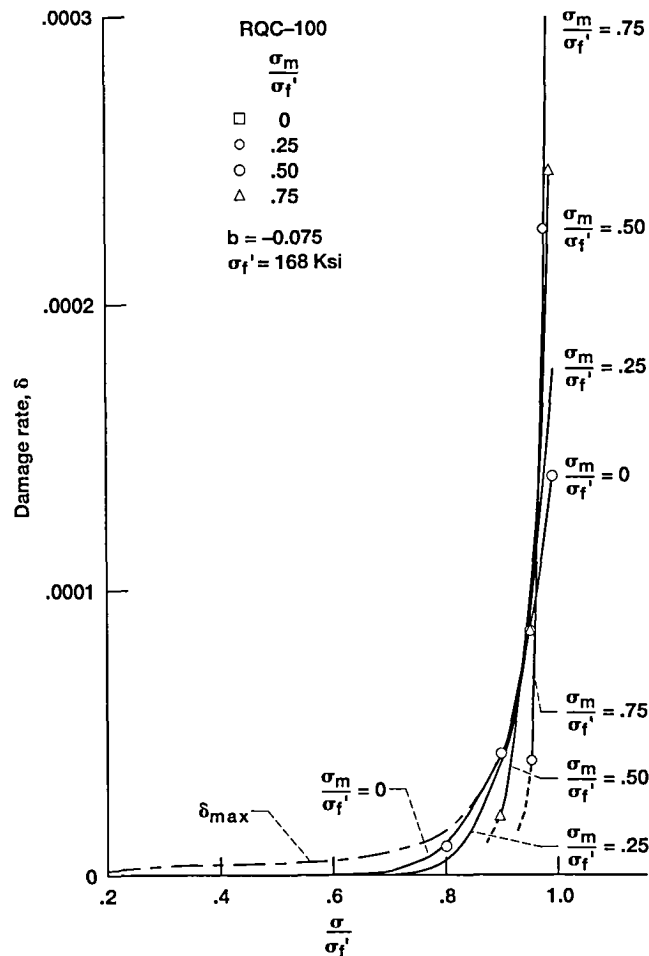


Figure 13.—Damage rate versus normalized stress level with  $\sigma_m/\sigma'_f = 0, .25, .50, .75$ .

predicted by equation 54. Note, if  $\sigma_m$  is allowed to be negative damage rates can exceed the zero mean stress damage prediction.

## Applications

### Controls Applications

The original intention of this work was to provide a simple continuum fatigue damage model for use in damage mitigating (for life extending) control. Two areas of application are possible; control of accumulated damage thru transients and secondly the avoidance of steady state operation in operating domains of high damage (when possible). To be useful for controls the primary requirement for a fatigue damage model is that it is continuum based (as opposed to cyclic), beyond this it must properly indicate at least approximately the magnitude of damage with respect to some primary variable (here stress). That is to say that the damage model must always indicate the correct direction to move the system to reduce damage however, the exact magnitude of the system movement (thru high model accuracy) may not be so important.

The damage models derived here (eqs. (18), (19), and (20)) are simple in that they are dependent only on the local stress level. A step more complex are the mean stress based models (eqs. (49), (50), and (51)). Here both stress and mean stress must be known to estimate damage rate. The ultimate in simplicity is the maximum damage envelopes model (eq. (54)). It depends only on stress and will always yield a conservative result and provide a proper direction for Life Extending Control.

### Design Applications

Availability of simple continuum fatigue models allows the multidisciplinary optimization of hardware design and system operation for long life. This concept has been forwarded for Reusable Rocket Engines (ref. 9) where it is proposed to optimize both aspects simultaneously. For design and operation optimization, again simplicity of computation is important since many computer iterations will be necessary. Use of a conservative model such as equation (54) could be used to closely approach an optimum then for final refinements more accurate (less conservative) model could be substituted to complete the optimization.

### Prognostic Applications

The determinations of critical component health could be done using the strain-strain rate model (eqs. (31) and (35)), it is believed (not verified) that this would provide more accurate results than the stress based models however, both stress and strain information would be required and the forms are considerably more complex than those that are stress based. The conservative forms (eqs. (20), (51), and (54)) may be used directly and could provide conservative prognosis of life/impending failure. Further it may be possible to calibrate these models for dedicated application to improve accuracy while maintaining their simplicity. It is noted that for repetitive cyclic damage these forms will be as accurate as the local strain method on which they were based. Use of the non-zero mean stress damage models (eqs. (49), (50), and (51)) pose the additional burden of determination of the mean stress  $\sigma_m$  for each cycle. For this reason use of the maximum damage rate form (eq. (54)) may be preferred even though it is conservative.

## Concluding Remarks

This effort seeks to create a continuum fatigue damage model. This initial effort attempts to mathematically convert the classic cycle based damage results as represented by the Local Strain Approach into a continuum model. The results achieved to date have been encouraging. Under the assumption of zero mean stress it has been shown that the continuum damage rate can be expressed as a highly nonlinear function of the instantaneous stress (eqs. (18) to (22)). These continuum rates are compatible with hysteretic damage. In the process of achieving these results explicit simplified forms of the cyclic damage laws have been developed (eqs. (15) and (46)). Also time domain implementations have been derived (eqs. (22) and (23)). A continuum fatigue damage model in terms of plastic

strain and plastic strain rate has been shown to be equivalent to the stress based model for zero mean stress hysteretic damage but is believed to be superior for general application. This has not been validated.

The case of non-zero mean stress is more difficult both conceptually and analytically. Here also, an explicit simplified form of the cyclic damage law (eq. (46)) has been developed. Based on this explicit form, stress level based damage rate expressions (eqs. (49) to (51)) have been derived under various assumptions. The assumptions center around broad questions of where in the cycle damage occurs.

Initial comparisons between the positive mean stress case and the zero mean stress case have shown that for stress levels up to 92.5 percent of  $\sigma'_f$  that the zero mean stress continuum model yields conservative results (i.e., greater damage). Thus for Life Extending (or Damage Mitigating) Control it may be applied to most practical situations.

Left open is the question of the nonlinear damage effects, namely the effect of accumulated damage on damage rate. While not reported yet, some progress has been made in this area with the availability of the simplified expressions discussed above. What is needed are appropriate cyclic data sets showing variations in stress amplitude versus cycles, for constant strain amplitude testing, to allow calibration of the effect. Further research effort is also needed to create plastic strain/strain-rate damage model for the mean stress case. Further, study of the relationship between this model and that for the zero mean stress should be done.

Finally experimental studies are required to screen broadly where the intracycle damage occurs. Such studies are now being started. Availability of such data will determine which assumptions of the analysis (i.e., tension damage only, etc.) apply. In the longer run further experimental validation of the proposed continuum models will also be required.

#### Acknowledgment

The author gratefully acknowledges benefits of discussion with Dr. Michael McGaw of NASA Lewis Research Center.

#### References

1. Lorenzo, C.F.; and Merrill, W.C.: Life Extending Control—A Concept Paper. June 26–28, 1991, Paper TA1: 1080, 1991 American Control Conference, Boston, MA, NASA TM-104391.
2. Ray, A.; Wu, M.-K.; Carpino, M., and Lorenzo, C.F.: Damage-Mitigating Control of Mechanical Systems: Part I—Conceptual Development and Model Formulation. American Control Conference, San Francisco, CA, June 1993; an expanded version to appear in Journal of Dynamic Systems, Measurement, and Control.
3. Ray, A.; Wu, M.-K.; Carpino, M.; and Lorenzo, C.F.: Damage-Mitigating Control of Mechanical Systems: Part II—Formulation of an Optimal Policy and Simulation. American Control Conference, June 1993; an expanded version to appear in Journal of Dynamic Systems, Measurement, and Control.
4. Ray, A.; Dai, X.; Wu, M.-K.; Carpino, M.; and Lorenzo, C.F.: Damage Mitigating Control of a Reusable Rocket Engine. Journal of Propulsion and Power, Vol. 10, No. 2, Mar.–Apr. 1994.
5. Ray, A.; and Wu, M.-K.: Damage Mitigating Control of Space Propulsion Systems for High Performance and Extended Life. NASA CR-194470, Mar. 1994.
6. Dowling, N.E.; Brose, W.R.; and Wilson, W.K.: Notched Member Fatigue Life Predictions by the Local Strain Approach. Fatigue Under Complex Loading: Analyses and Experiments, Library of Congress CCN 77-85454 Society of Automotive Engineers, Inc., 1977.

7. Lorenzo, C.F.: Continuum Fatigue Damage Modeling for Use in Life Extending Control. NASA TM-106691, Aug. 1994.

8. Dowling, N.E.: Fatigue Life Prediction for Complex Load Versus Time Histories. Journal of Engineering Materials and Technology, Trans. ASME, Vol. 105, 1983, pp. 206-214.

9. Lorenzo, C.F.: Robust Rocket Engine Concept. NASA TM-107010, 31st Joint Propulsion Conference, AIAA, ASME, SAE, and ASEE, San Diego, CA, July 1995.

REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188	
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.				
1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE January 1996	3. REPORT TYPE AND DATES COVERED Technical Memorandum		
4. TITLE AND SUBTITLE Continuum Fatigue Damage Modeling for Critical Design, Control, and Fault Prognosis		5. FUNDING NUMBERS  WU-505-62-05		
6. AUTHOR(S)  Carl F. Lorenzo				
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)  National Aeronautics and Space Administration Lewis Research Center Cleveland, Ohio 44135-3191		8. PERFORMING ORGANIZATION REPORT NUMBER  E-9926		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)  National Aeronautics and Space Administration Washington, D.C. 20546-0001		10. SPONSORING/MONITORING AGENCY REPORT NUMBER  NASA TM-107065		
11. SUPPLEMENTARY NOTES Prepared for the Sixth International Symposium on Transport Phenomena and Dynamics of Rotating Machinery cosponsored by the Pacific Center of Thermal Fluids Engineering and the U.S. Turbo and Power Machinery Research Center, Honolulu, Hawaii, February 25-29, 1996. Responsible person, Carl F. Lorenzo, organization code 2500, (216) 977-3733.				
12a. DISTRIBUTION/AVAILABILITY STATEMENT  Unclassified - Unlimited Subject Categories 39 and 37  This publication is available from the NASA Center for Aerospace Information, (301) 621-0390.			12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words)  This paper develops a simplified continuum (continuous with respect to time, stress, etc.) fatigue damage model for use in critical design, Life Extending Control and fault prognosis. The work is based on the local strain cyclic damage modeling method. New nonlinear explicit equation forms of cyclic damage in terms of stress amplitude are derived to facilitate the continuum modelling. Stress based continuum models are derived. Extension to plastic strain—strain rate models is also presented. Progress toward a non-zero mean stress based is presented. Also new nonlinear explicit equation forms in terms of stress amplitude are derived for this case. Application of the various models to design, control, and fault prognosis is considered.				
14. SUBJECT TERMS  Continuum fatigue damage modelling; Critical design; Life extending control			15. NUMBER OF PAGES 24	
			16. PRICE CODE A03	
17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT Unclassified	20. LIMITATION OF ABSTRACT	